SYMMETRIC MONOCHROMATIC SUBSETS IN COLORINGS OF THE LOBACHEVSKY PLANE

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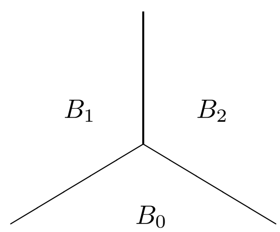
Abstract. We prove that for each partition of the Lobachevsky plane into finitely many Borel pieces one of the cells of the partition contains an unbounded centrally symmetric subset.

It follows from [B_1] (see also [BP_1, Theorem 1]) that for each partition of the n-dimensional space $\mathbb{R}^n$ into n pieces one of the pieces contains an unbounded centrally symmetric subset. On the other hand, $\mathbb{R}^n$ admits a partition into $(n+1)$ Borel pieces containing no unbounded centrally symmetric subset. For $n=2$ such a partition is drawn at the picture:



Taking the same partition of the Lobachevsky plane H^2, we can see that each cell B_i does contain a unbounded centrally symmetric subset (for such a set just take any hyperbolic line lying in B_i). We call a subset S of the hyperbolic plane H^2 *centrally symmetric or else symmetric with respect to a point* $c \in H^2$ if $S = f_c(S)$ where $f_c : H^2 \to H^2$ is the involutive isometry of H^2 assigning to each point $x \in H^2$ the unique point $y \in H^2$ such that c is the midpoint of the segment $[x, y]$. The map f_c is called the *central symmetry* of H^2 with respect to the point c.

The following theorem shows that the Lobachevsky plane differs dramatically from the Euclidean plane from the Ramsey point of view.

Theorem 1. *For any partition $H^2 = B_1 \cup \dots \cup B_m$ of the Lobachevsky plane into finitely many Borel pieces one of the pieces contains an unbounded centrally symmetric subset.*

Proof. We shall prove a bit more: given a partition $H^2 = B_1 \cup \dots \cup B_m$ of the Lobachevsky plane into m Borel pieces we shall find $i \le m$ and an unbounded subset $S \subset B_i$ symmetric with respect to some point c in an arbitrarily small neighborhood of some finite set $F \subset H^2$ depending only on m.

To define this set F it will be convenient to work in the Poincaré model of the Lobachevsky plane H^2. In this model the hyperbolic plane H^2 is identified with

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the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane and hyperbolic lines are just segments of circles orthogonal to the boundary of $\mathbb{D}$. Let $\overline{\mathbb{D}} = \{z \in \mathbb{C} : |z| \le 1\}$ be the hyperbolic plane $\mathbb{D}$ with attached ideal line. For a real number $R > 0$ the set $\mathbb{D}_R = \{z \in \mathbb{C} : |z| \le 1 - 1/R\}$ can be thought as a hyperbolic disk of increasing radius as R tends to ∞.

On the boundary of the unit disk $\mathbb{D}$ consider the $(m+1)$-element set

$$A = \{z \in \mathbb{C} : z^{m+1} = 1\}.$$

For any two distinct points $x, y \in A$ by $[x|y] \in \mathbb{D}$ we denote the "Euclidean" midpoint of the arc in $\overline{\mathbb{D}}$ that connects the points x, y and lies on a hyperbolic line in $H^2 = \mathbb{D}$. Then $F = \{[x|y] : x, y \in A,\ x \ne y\}$ is a finite subset of cardinality $|F| \le m(m+1)/2$ in the unit disk $\mathbb{D}$.

We claim that for any open neighborhood W of F in $\mathbb{C}$ one of the cells of the partition $H^2 = B_1 \cup \dots \cup B_m$ contains an unbounded subset symmetric with respect to some point $c \in W$. To derive a contradiction we assume the converse and conclude that for every point $c \in W$ and every $i \le m$ the set $B_i \cap f_c(B_i)$ is bounded in H^2.

For every $n \in \mathbb{N}$ consider the set

$$C_n = \{c \in W : \bigcup_{i=1}^{m} B_i \cap f_c(B_i) \subset \mathbb{D}_n\}.$$

We claim that C_n is a coanalytic subset of W. The latter means that the complement $W \setminus C_n$ is analytic, i.e., is the continuous image of a Polish space. Observe that

$$W \setminus C_n = \{c \in W : \exists i \le m\ \exists x \in \mathbb{D} \setminus \mathbb{D}_n,\ x \in B_i \text{ and } x \in f_c(B_i)\} = \mathrm{pr}_2(E)$$

where $\mathrm{pr}_2 : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ is the projection on the second factor and

$$E = \bigcup_{i=1}^{m} \{(x, c) \in \mathbb{D} \times W : x \in \mathbb{D} \setminus \mathbb{D}_n,\ x \in B_i \text{ and } f_c(x) \in B_i\}$$

is a Borel subset of $\mathbb{D} \times W$. Being a Borel subset of the Polish space $\mathbb{D} \times W$, the space E is analytic and so is its continuous image $\mathrm{pr}_2(E) = W \setminus C_n$. Then C_n is coanalytic and hence has the Baire property [Ke, 21.6], which means that C_n coincides with an open subset U_n of W modulo some meager set. The latter means that the symmetric difference $U_n \triangle C_n$ is meager (that is, of the first Baire category in W). Since $C_n \subset C_{n+1}$, we may assume that $U_n \subset U_{n+1}$ for all $n \in \mathbb{N}$. Let $U = \bigcup_{n=1}^{\infty} U_n$ and $M = \bigcup_{n=1}^{\infty} U_n \triangle C_n$.

Taking into account that $W = \bigcup_{n=1}^{\infty} C_n$, we conclude that

$$W \setminus U = \bigcup_{n=1}^{\infty} C_n \setminus \bigcup_{n=1}^{\infty} U_n \subset \bigcup_{n=1}^{\infty} C_n \setminus U_n \subset \bigcup_{n=1}^{\infty} C_n \triangle U_n = M$$

which implies that the open set U has meager complement and thus is dense in W.

We claim that $F \subset h^{-1}(U)$ for some isometry h of the hyperbolic plane $H^2 = \mathbb{D}$.

For this consider the natural action

$$\mu : \mathrm{Iso}(H^2) \times \mathbb{D} \to \mathbb{D},\ \ \mu : (h, x) \mapsto h(x)$$

of the isometry group $\mathrm{Iso}(H^2)$ of the hyperbolic plane $H^2 = \mathbb{D}$. It is easy to see that for every $x \in \mathbb{D}$ the map $\mu_x : \mathrm{Iso}(H^2) \to \mathbb{D}$, $\mu_x : h \mapsto h(x)$, is continuous and

open (with respect to the compact-open topology on $\mathrm{Iso}(H^2)$). It follows that the set

$$\bigcap_{x\in F}\mu_x^{-1}(W)=\{h\in \mathrm{Iso}(H^2):f(F)\subset W\}$$

is an open neighborhood of the neutral element of the group $\mathrm{Iso}(H^2)$.

Taking into account that U is open and dense in W and for every $x\in F$ the map $\mu_x:\mathrm{Iso}(H^2)\to\mathbb{D}$ is open, we conclude that the preimage the set $\mu_x^{-1}(U)$ is open and dense in $\mu_x^{-1}(W)\subset \mathrm{Iso}(H^2)$. Then the intersection $\bigcap_{x\in F}\mu_x^{-1}(U)$, being an open dense subset of $\bigcap_{x\in F}\mu_x^{-1}(W)$, is not empty and hence contains some isometry h having the desired property: $F\subset h^{-1}(U)$. Since F is finite, there is $R\in\mathbb{N}$ with $F\subset h^{-1}(U_R)$. For a complex number $r\in\mathbb{D}$ consider the set $rA=\{rz:z\in A\}\subset\mathbb{D}$ and let

$$F_r=\{[x|y]:x,y\in rA,\ x\ne y\}\subset\mathbb{D},$$

where $[x|y]$ stands for the midpoint of the hyperbolic segment connecting x and y in H^2. It can be shown that for any distinct points $x,y\in A$ the “hyperbolic” midpoint $[rx|ry]$ tends to the “Euclidean” midpoint $[x|y]$ as r tends to 1. Such a continuity yields a neighborhood O_1 of 1 such that $F_r\subset h^{-1}(U_R)$ for all $r\in O_1\cap\mathbb{D}$.

It is clear that for any points $x,y\in A$ the map

$$f_{x,y}:\mathbb{D}\to\mathbb{D},\ f_{x,y}:r\mapsto[rx|ry]$$

is open and continuous. Consequently, the preimage $f_{x,y}^{-1}(h^{-1}(M))$ is a meager subset of $\mathbb{D}$ and so is the union $M'=\bigcup_{x,y\in A}f_{x,y}^{-1}(h^{-1}(M))$. So, we can find a non-zero point $r\in O_1\setminus M'$ so close to 1 that the set rA is disjoint with the hyperbolic disk $h^{-1}(\mathbb{D}_R)$. For this point r we shall get $F_r\cap h^{-1}(M)=\emptyset$.

The set rA consists of $m+1$ points. Consequently, some cell $h^{-1}(B_i)$ of the partition $\mathbb{D}=h^{-1}(B_1)\cup\dots\cup h^{-1}(B_m)$ contains two distinct points rx,ry of rA. Those points are symmetric with respect to the point

$$[rx|ry]\in F_r\subset h^{-1}(U_R)\setminus h^{-1}(M).$$

Then the images $a=h(rx)$ and $b=h(ry)$ belong to B_i and are symmetric with respect to the point $c=h([rx|ry])\in U_R\setminus M\subset C_R$. It follows from the definition of C_R that $\{a,b\}\subset B_i\cap f_c(B_i)\subset\mathbb{D}_R$, which is not the case because $rx,ry\notin h^{-1}(\mathbb{D}_R)$. □

We do not know if Theorem 1 is true for any finite (not necessarily Borel) partition of the Lobachevsky plane H^2. For partitions of H^2 into two pieces the Borel assumption is supefluous.

Theorem 2. *There is a subset $T\subset H^2$ of cardinality $|T|=3$ such that for any partition $H^2=A_1\cup A_2$ of H^2 into two pieces either A_1 or A_2 contains an unbounded subset, symmetric with respect to some point $c\in T$.*

Proof. Lemma 1 below allows us to find an equilateral triangle $\triangle c_0c_1c_2$ on the Lobachevsky plane H^2 such that the composition $f_{c_2}\circ f_{c_1}\circ f_{c_0}$ of the symmetries with respect to the points c_0,c_1,c_2 coincides with the rotation on the angle $2\pi/3$ around some point $o\in H^2$. Consequently $(f_{c_2}\circ f_{c_1}\circ f_{c_0})^3$ is the identity isometry of H^2.

We claim that for any partition $H^2=A_1\sqcup A_2$ of the Lobachevsky plane into two pieces one of the pieces contains an unbounded subset symmetric with respect

to some point in the triangle $T = \{c_0, c_1, c_2\}$. Assuming the converse, we conclude that the set

$$B = \bigcup_{c \in T} \bigcup_{i=1}^{2} A_i \cap f_c(A_i)$$

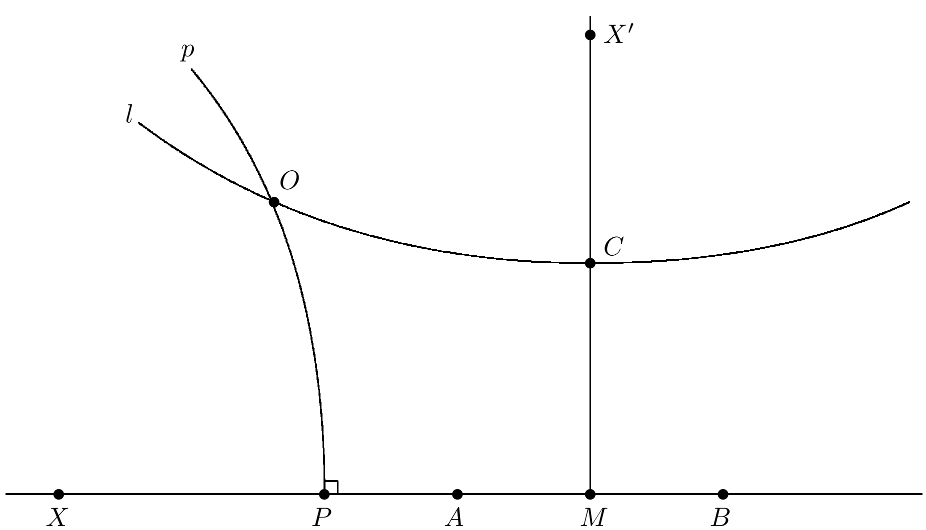
is bounded. It follows that two points $x, y \in H^2 \setminus B$, symmetric with respect to a center $c \in T$ cannot belong to the same cell A_i of the partition.

Given a point $x_0 \in H^2$ consider the sequence of points $x_1, \dots x_9$ defined by the recursive formula: $x_{i+1} = f_{c_{i \bmod 3}}(x_i)$. It follows that $x_9 = (f_{c_2} \circ f_{c_1} \circ f_{c_0})^3(x_0) = x_0$. Taking x_0 sufficiently far from the center o of rotation we can guarantee that none of the points $x_0, \dots, x_9$ belongs to B.

The point x_0 belongs either to A_1 or to A_2. We lose no generality assuming that $x_0 \in A_2$. Since the points $x_0, x_1 \notin B$ are symmetric with respect to c_0 and $x_0 \in A_2$, we get that $x_1 \in H^2 \setminus A_2 = A_1$. By the same reason x_1, x_2 cannot simultaneously belong to A_1 and hence $x_2 \in A_2$. Continuing in this fashion we conclude that x_i belongs to A_1 for odd i and to A_2 for even i. In particular, $x_9 \in A_1$, which is not possible because $x_9 = x_0 \in A_2$. □

Lemma 1. *There is an equilateral triangle $\triangle ABC$ on the Lobachevsky plane such that the composition $f_C \circ f_B \circ f_A$ of the symmetries with respect to the points A, B, C coincides with the rotation on the angle $2\pi/3$ around some point O.*

Proof. For a positive real number t consider an equilateral triangle $\triangle ABC$ with side t the on the Lobachevsky plane. Let M be the midpoint of the side AB and l be the line through C that is orthogonal to the line CM. Consider also the line p that is orthogonal to the line AB and passes through the point P such that A is the midpoint between P and M. Observe that $|PM| = |AB| = t$ and for sufficiently small t the lines p and l intersect at some point O.



It is easy to see that the composition $f_B \circ f_A$ is the shift along the line AB on the distance $2t$ and hence the image $f_B \circ f_A(O)$ of the point O is the point symmetric to O with respect to the point C. Consequently, $f_C \circ f_B \circ f_A(O) = O$, which means

that the isometry $f_C \circ f_B \circ f_A$ is a rotation of the Lobachevsky plane around the point O on some angle φ_t.

To estimate this angle, consider the point X such that P is the midpoint between X and M. Then $|XM| = 2t$ and consequently, $f_B \circ f_A(X) = M$ while $X' = f_C \circ f_B \circ f_A = f_C(M)$ is the point on the line CM such that C is the midpoint between X' and M. It follows that $|X'X| \le |XM| + |MX'| < 2t + 2t = 4t$.

Observe that for small t the point X' is near to the point, symmetric to X with respect to O, which means that the angle $\varphi_t = \angle XOX'$ is close to π for t close to zero. On the other hand, for very large t the lines p and l on the Lobachevsky plane do not intersect. So we can consider the smallest upper bound t_0 of numbers t for which the lines l and p meet. For values $t < t_0$ near to t_0 the point O tends to infinity as t tends to t_0. Since the length of the side XX' of the triangle $\triangle XOX'$ is bounded by $4t_0$ the angle $\varphi_t = \angle XOX'$ tends to zero as O tends to infinity. Since the angle φ_t depends continuous on t and decreases from π to zero as t increases from zero to t_0, there is a value t such that $\varphi_t = 2\pi/3$. For such t the composition $f_C \circ f_B \circ f_A$ is the rotation around O on the angle $2\pi/3$. □

Some comments and Open Problems

In contrast to Theorem 1, Theorem 2 is true for the Euclidean plane E^2 even in a stronger form: for any subset $C \subset E^2$ not lying on a line and any partition $E^2 = A_1 \cup A_2$ one of the cells of the partition contains an unbounded subset symmetric with respect to some center $c \in C$, see [B$_2$].

Having in mind this result let us call a subset C of a Lobachevsky or Euclidean space X *central for (Borel) k-partitions* if for any partition $X = A_1 \cup \dots \cup A_k$ of X into k (Borel) pieces one of the pieces contains an unbounded monochromatic subset $S \subset X$, symmetric with respect to some point $c \in C$. By $c_k(X)$ (resp. $c_k^B(X)$) we shall denote the smallest size of a subset $C \subset X$, central for (Borel) k-partitions of X. If no such a set C exists, then we put $c_k(X) = \infty$ (resp. $c_k^B(X) = \infty$) where ∞ is assumed to be greater than any cardinal number. It follows from the definition that $c_k^B(X) \le c_k(X)$.

We have a lot of information on the numbers $c_k^B(E^n)$ and $c_k(E^n)$ for Euclidean spaces E^n, see [B$_2$]. In particular, we known that

1. $c_2(E^n) = c_2^B(E^n) = 3$ for all $n \ge 2$;
2. $c_3(E^3) = c_3^B(E^3) = 6$;
3. $12 \le c_4^B(E^4) \le c_4(E^4) \le 14$;
4. $n(n+1)/2 \le c_n^B(E^n) \le c_n(E^n) \le 2^n - 2$ for every $n \ge 3$.

Much less is known on the numbers $c_k^B(H^n)$ and $c_k^B(H^n)$ in the hyperbolic case. Theorem 2 yields the upper bound $c_2(H^2) \le 3$. In fact, 3 is the exact value of $c_2(H^n)$ for all $n \ge 2$.

Proposition 1. *$c_2^B(H^n) = c_2(H^n) = 3$ for all $n \ge 2$.*

Proof. The upper bound $c_2(H^n) \le c_2(H^2) \le 3$ follows from Theorem 2. The lower bound $3 \le c_2^B(H^n)$ will follow as soon as for any two points $c_1, c_2 \in H^n$ we construct a partition $H^n = A_1 \cup A_2$ in two Borel pieces containing no unbounded set, symmetric with respect to a point c_i. To construct such a partition, consider the line l containing the points c_1, c_2 and decompose l into two half-lines $l = l_1 \sqcup l_2$. Next, let H be an $(n-1)$-hyperplane in H^n, orthogonal to the line l. Let S be the unit sphere in H centered at the intersection point of l and H. Let $S = B_1 \cup B_2$

be a partition of S into two Borel pieces such that no antipodal points of S lie in the same cell of the partition. For each point $x \in H^n \setminus l$ consider the hyperbolic plane P_x containing the points x, c_1, c_2. The complement $P_x \setminus l$ decomposes into two half-planes $P_x^+ \cup P_x^-$ where P_x^+ is the half-plane containing the point x. The plane P_x intersects the hyperplane H by a hyperbolic line containing two points of the sphere S. Finally put

$$A_i = l_i \cup \{x \in H^2 \setminus l : P_x^+ \cap B_i \neq \emptyset\}$$

for $i \in \{1, 2\}$. It is easy to check that $A_1 \sqcup A_2 = H^n$ is the desired partition of the hyperbolic space into two Borel pieces none of which contains an unbounded subset symmetric with respect to one of the points c_1, c_2. □

The preceding proposition implies that the cardinal numbers $c_2(H^n)$ are finite.

Problem 1. *For which numbers k, n are the cardinal numbers $c_k(H^n)$ and $c_k^B(H^n)$ finite? Is it true for all $k \le n$?*

Except for the equality $c_2(E^n) = 3$, we have no information on the numbers $c_k(E^n)$ with $k < n$.

Problem 2. *Calculate (or at least evaluate) the numbers $c_k(E^n)$ and $c_k(H^n)$ for $2 < k < n$.*

In all the cases where we know the exact values of the numbers $c_k(E^n)$ and $c_k^B(E^n)$ we see that those numbers are equal.

Problem 3. *Are the numbers $c_k(E^n)$ and $c_k^B(E^n)$ (resp. $c_k(H^n)$ and $c_k^B(H^n)$) equal for all k, n?*

Having in mind that each subset not lying on a line is central for 2-partitions of the Euclidean plane, we may ask about the same property of the Lobachevsky plane.

Problem 4. *Is any subset $C \subset H^2$ not lying on a line central for (Borel) 2-partitions of the Lobachevsky plane H^2?*

Finally, let us ask about the numbers $c_k^B(H^2)$ and $c_k(H^2)$. Observe that Theorem 1 guarantees that $c_k^B(H^2) \le \mathfrak{c}$ for all $k \in \mathbb{N}$. Inspecting the proof we can see that this upper bound can be improved to $c_k^B(H^2) \le \mathrm{non}(\mathcal{M})$ where $\mathrm{non}(\mathcal{M})$ is the smallest cardinality of a non-meager subset of the real line. It is clear that $\aleph_1 \le \mathrm{non}(\mathcal{M}) \le \mathfrak{c}$. The exact location of the cardinal $\mathrm{non}(M)$ on the interval $[\aleph_1, \mathfrak{c}]$ depends on axioms of Set Theory, see [Bl]. In particular, the inequality $\aleph_1 = \mathrm{non}(\mathcal{M}) < \mathfrak{c}$ is consistent with ZFC.

Problem 5. *Is the inequality $c_k^B(H^2) \le \aleph_1$ provable in ZFC? Are the cardinals $c_k^B(H^2)$ countable? finite?*

The latter problem asks if H^2 contains a countable (or finite) central set for Borel k-partitions of the Lobachevsky plane. Inspecting the proof of Theorem 1 we can see that it gives an "approximate" answer to this problem:

Proposition 2. *For any $k \in \mathbb{N}$ there is a finite subset $C \subset H^2$ of cardinality $|C| \le k(k+1)/2$ such that for any partition $H^2 = B_1 \cup \cdots \cup B_k$ of H^2 into k Borel pieces and for any open neighborhood $O(C) \subset H^2$ of C one of the pieces B_i contains an unbounded subset $S \subset B_i$ symmetric with respect to some point $c \in O(C)$.*

Remark 1. For further results and open problems related to symmetry and colorings see the surveys [BP_2], [BVV] and the list of problems [BBGRZ, §4].

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